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COMMENT

Toda lattice and generalised Wronskian technique†

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Abstract. By an extension of the generalised Wronskian technique we tersely reobtain the Bäcklund transformations and the class of nonlinear discrete evolution equations whose first member is the Toda lattice.

In a recent paper, Dodd (1978) derived the Bäcklund transformation and other functional relations connected to the Toda lattice by using the scattering problem satisfied by a generalisation of the 'squared eigenfunctions', following Ablowitz *et al* (1974). In this paper we show that by a no more complex procedure these same results can be obtained by the generalised Wronskian technique introduced by Calogero (1975).

For the basic notation and the discussion of the main results we refer to Dodd (1978).

Let us consider the scattering problem

$$a(n-1)u(n-1)+a(n)u(n+1)+b(n)u(n)=\lambda u(n). \tag{1}$$

We define the Wronskian between two solutions $u(n)$, $\bar{u}(n)$ of (1) as

$$W[u(n),\bar{u}(n)]=u(n+1)a(n)\bar{u}(n)-\bar{u}(n+1)a(n)u(n), \tag{2}$$

which turns out to be independent of n.

To the same spectrum we associate two different eigenvalue problems (1), characterised by the potentials $(a(n), b(n))$, $(a'(n), b'(n))$ and by their corresponding eigenfunctions $u(n)$, $u'(n)$; for these two spectral problems we define the generalised Wronskian as follows:

$$\tilde{W}[u(n),\tilde{u}(n)]=u(n+1)a'(n)\tilde{u}(n)-\tilde{u}(n+1)a(n)u(n), \tag{3a}$$

where

$$\tilde{u}(n)=c(n)f(n)u'(n)+c^{-1}(n-1)a'(n)g(n)u'(n+1), \tag{3b}$$

$c(n)$ being defined as

$$c(n)=\prod_{j=-\infty}^{n}\frac{a(j)}{a'(j)} \tag{4}$$

and $f(n)$, $g(n)$ being *arbitrary* functions of n.

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Performing a variation of the generalised Wronskian ($3a$) between two neighbouring lattice points and then summing from $-\infty$ to $+\infty$, we obtain

$$
\begin{aligned}
&[u(n+1)a'(n)c(n)f(n)u'(n)-u(n)a(n)a^{-1}(n+1)a'(n+1)\\
&\quad\times c(n+1)f(n+1)u'(n+1)+u(n+1)a'(n)a(n)c^{-1}(n-1)g(n)u'(n+1)\\
&\quad-u(n)a(n)a'(n+1)c^{-1}(n)g(n+1)u'(n+2)]_{-\infty}^{+\infty}\\
&\quad=\lambda\sum_{n=-\infty}^{+\infty}\{v(n)c(n-1)[f(n)-f(n-1)]+w(n)a'(n)c^{-1}(n-1)[g(n)-g(n+1)]\}\\
&\quad-\sum_{n=-\infty}^{+\infty}[\![v(n)\{c^{-1}(n-1)[a^2(n-1)g(n-1)-a'^2(n)g(n+1)]\\
&\quad-c(n-1)[b'(n)f(n-1)-b(n)f(n)]\}\\
&\quad+w(n)a'(n)\{c(n-1)[a^2(n)a'^{-2}(n)f(n+1)-f(n-1)]\\
&\quad+c^{-1}(n-1)[b(n)g(n)-b'(n+1)g(n+1)]\}]\!]
\end{aligned}
\tag{5}
$$

where, following Dodd (1978), we have set

$$v(n)=u(n)u'(n),\qquad w(n)=u(n)u'(n+1).$$

Equation (5) is the main result of the generalised Wronskian technique, and from it we are able to derive all relevant results, due to the arbitrariness of $f(n)$, $g(n)$.

Choosing $f(n)=f$, $g(n)=g$ (f and g being two arbitrary constants), we obtain

$$
\begin{aligned}
&f\alpha\alpha'\bigg[c(+\infty)\mathrm{i}\sin k\Big(\frac{\beta}{\alpha}-\frac{\beta'}{\alpha'}\Big)-\sum_{n=-\infty}^{+\infty}[v(n)c(n-1)(b'(n)-b(n))\\
&\quad+w(n)c(n-1)a'(n)(1-a^2(n)a'^{-2}(n))]\bigg]+g\alpha\alpha'\bigg[c^{-1}(+\infty)\mathrm{i}\sin k\\
&\quad\times\Big(\frac{\beta}{\alpha}\mathrm{e}^{-\mathrm{i}k}-\frac{\beta'}{\alpha'}\mathrm{e}^{\mathrm{i}k}\Big)-\sum_{n=-\infty}^{+\infty}[v(n)c^{-1}(n-1)(a'^2(n)-a^2(n-1))\\
&\quad+w(n)a'(n)c^{-1}(n-1)(b'(n+1)-b(n))]\bigg]=0
\end{aligned}
\tag{6}
$$

and by choosing $f(n)$ and $g(n)$ such that they go to zero at $+\infty$ and to a constant value at $-\infty$ we obtain

$$
\begin{aligned}
&\lambda\sum_{n=-\infty}^{+\infty}[v(n)p(n)+w(n)q(n)]\\
&\quad=\sum_{n=-\infty}^{+\infty}\bigg[v(n)\bigg(a(n-1)q(n-1)+a^2(n-1)a'^{-1}(n)q(n)+b'(n)p(n)\\
&\quad+(a^2(n-1)-a'^2(n))c^{-1}(n-1)\sum_{j=n+1}^{\infty}c(j-1)a'^{-1}(j)q(j)\\
&\quad+(b'(n)-b(n))c(n-1)\sum_{j=n+1}^{\infty}c^{-1}(j-1)p(j)\bigg)\\
&\quad+w(n)\bigg(a'(n)p(n)+a(n)p(n+1)+b(n)q(n)\\
&\quad+(a'^2)(n)-a^2(n))a'^{-1}(n)c(n-1)\sum_{j=n+1}^{\infty}c^{-1}(j-1)p(j)\\
&\quad+(b(n)-b'(n+1))a'(n)c^{-1}(n-1)\sum_{j=n+1}^{\infty}c(j-1)a'^{-1}(j)q(j)\bigg)\bigg]
\end{aligned}
\tag{7}
$$

where we have defined

$$p(n)=c(n-1)[f(n)-f(n-1)], \qquad q(n)=c^{-1}(n-1)a'(n)[g(n)-g(n+1)].$$

Equation (7) provides us with the exact form of the operator Λ^+ of Dodd (1978) while from equation (6), by choosing $g=0$, $f=1$ and $g=1$, $f=0$ we obtain two different functional expressions connecting the scattering data to the potentials which are the starting point for obtaining the Bäcklund transformations and the evolution equations (Dodd 1978, equations (3.5), (2.20*b*)).

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